Introduction to Linear Algebra and Differential Equations

Elijah Kin

February 3, 2026

Contents

1 Introduction

1.1 What are Differential Equations?

In order to define what a differential equation is, it is useful to have another concept to compare it against. An *algebraic equation* is something such as $x + 5 = 12$ or $y^2 - y + 1 = 0$; there is an unknown variable, and to "solve" the equation is to give a numerical value for the unknown variable which satisfies the equation.

In contrast, a *differential equation* is an equation involving an unknown function and one or more of its derivatives. To "solve" a differential equation is to give a function which satisfies the equation.

You may ask why we should study differential equations. The answer is that it is often easier to describe how a certain quantity is changing than the value of that quantity itself.

Example 1.1.1. Suppose we are interested in Maryland's rabbit population $P(t)$ as a function of time. With the (unreasonable) assumption of unlimited carrying capacity, then it is natural to expect that $P' \propto P$. In other words, the more rabbits there are, the more rabbits are being born at any given time. Introducing a constant of proportionality r, we can express this relationship as the differential equation

$$\frac{\mathrm{d}P}{\mathrm{d}t} = rP.$$

Can you solve this using what you know from calculus?

Example 1.1.2. Suppose I am holding a basketball 1 meter off the ground and throw it straight upwards with an initial velocity of 8 m/s. Recall that acceleration due to gravity is 9.8 $\mathrm{m/s^2}$. This means that the ball's distance from the ground $y(t)$ in meters as a function of time is governed by the differential equation

$$\frac{\mathrm{d}^2y}{\mathrm{d}t^2} = -9.8$$

with the initial condition $y(0) = 1$. Can you solve this? How high will the basketball get?

Example 1.1.3. The SIR model is popular for modeling the spread of infectious diseases. It compartmentalizes people into three categories: susceptible (S), infectious (I), and recovered (R). Initially, 1 person may be infectious, 0 people have recovered, and everyone else is susceptible. Then by making similar assumptions as in the rabbit example, we can predict the spread of the disease will be governed by the system of differential equations

$$\frac{\mathrm{d}S}{\mathrm{d}t} = -\frac{\beta}{N}IS \qquad \frac{\mathrm{d}I}{\mathrm{d}t} = \frac{\beta}{N}IS - \gamma I \qquad \frac{\mathrm{d}R}{\mathrm{d}t} = \gamma I$$

We will talk more about systems of differential equations at the end of the semester.

1.2 Calculus Review

It is crucial that you are comfortable with common integration methods, especially u-substitution and integration by parts.

Exercise 1.2.1. Evaluate $\displaystyle\int \frac{1}{t \ln t}\,\mathrm{d}t$.

Exercise 1.2.2. Evaluate $\displaystyle\int te^t\,\mathrm{d}t$.

2 First-Order Linear Differential Equations

2.1 Classification of Differential Equations

A *linear* differential equation is one in which the unknown function and its derivatives appear linearly; that is, terms may be scaled by functions of t and added together but no more, like so:

$$a_0(t)y + a_1(t)\frac{\mathrm{d}y}{\mathrm{d}t} + a_2(t)\frac{\mathrm{d}^2y}{\mathrm{d}t^2} + \cdots + a_n(t)\frac{\mathrm{d}^ny}{\mathrm{d}t^n} = f(t).$$

The *order* of a differential equation refers to the highest derivative present in the equation; hence, a first-order differential equation contains a first derivative term, but no higher derivatives. Our focus for the beginning of the semester will be on first-order linear differential equations, which can be written in the so-called *linear normal form*:

$$\frac{\mathrm{d}y}{\mathrm{d}t} + a(t)y = f(t).$$

When the function $f(t)$ is identically zero, we will call the differential equation *homogeneous*.

Exercise 2.1.1. For each differential equation below, determine its order and whether it is linear. If so, write it in linear normal form and determine whether it is homogeneous.

(a) $\dfrac{\mathrm{d}P}{\mathrm{d}t} = rP$, modeling population growth with unlimited carrying capacity.

(b) $\dfrac{\mathrm{d}P}{\mathrm{d}t} = rP\left(1 - \dfrac{P}{L}\right)$, modeling population growth with carrying capacity L.

(c) $m\dfrac{\mathrm{d}^2x}{\mathrm{d}t^2} = -kx$, governing simple harmonic oscillators such as a mass on a spring.

(d) $\dfrac{\mathrm{d}T}{\mathrm{d}t} = -k(T - T_a)$, Newton's law of cooling with ambient temperature T_a.

2.2 Solving Homogeneous Equations

We're now able to classify differential equations, so how can we solve them? We'll start by studying first-order homogeneous differential equations, which can be written in the linear normal form:

$$\frac{\mathrm{d}y}{\mathrm{d}t} + a(t)y = 0.$$

But what is this asking? Well, for a function $y(t)$ whose first derivative is $a(t)$ times itself. Recalling the chain rule and the properties of the exponential function, we can guess the family of solutions

$$y(t) = Ce^{A(t)} \text{ where } A(t) = \int -a(t)\,\mathrm{d}t\,.$$

Note that you are always able to check the solution to a differential equation by differentiating and ensuring the equation is satisfied. The above is a "family" of solutions because the value of C can be any real number.

Exercise 2.2.1. Solve the first-order homogeneous differential equation $\dfrac{\mathrm{d}y}{\mathrm{d}t} = t^2y$ with $y(0) = 5$.

2.3 Integrating Factors

We now know how to solve homogeneous equations, but what about the general linear normal form

$$\frac{\mathrm{d}y}{\mathrm{d}t} + a(t)y = f(t)$$

where $f(t)$ is not necessarily zero? For this we will use *integrating factors*, which are based on the following idea: let $A(t)$ be an antiderivative of $a(t)$ and notice that by the product rule,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[e^{A(t)}y\right] = e^{A(t)}\frac{\mathrm{d}y}{\mathrm{d}t} + a(t)e^{A(t)}y = \left(\frac{\mathrm{d}y}{\mathrm{d}t} + a(t)y\right)e^{A(t)},$$

which we recognize as $e^{A(t)}$ times the left-hand side of our differential equation. Hence, multiplying our differential equation by $e^{A(t)}$, we can rewrite it as

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[e^{A(t)}y\right] = e^{A(t)}f(t),$$

which we will call the *integrating factor form* of the differential equation. Then by integrating both sides, we can obtain the family of solutions

$$y(t) = e^{-A(t)}\int e^{A(t)}f(t)\,\mathrm{d}t + Ce^{-A(t)}.$$

Exercise 2.3.1. A 120 gal tank initially contains 90 lb of salt dissolved in 90 gal. Brine containing 2 lb/gal of salt flows into the tank at a rate of 4 gal/min, and the well-stirred mixture flows out of the tank at a rate of 3 gal/min. How much salt does the tank contain when it is full?

Solution: The salt in the tank is changing according to the differential equation

$$\frac{\mathrm{d}S}{\mathrm{d}t} = S_{\text{in}} - S_{\text{out}} = (2\cdot 4) - \left(\frac{S}{90+t}\cdot 3\right) = 8 - \frac{3S}{90+t}$$

with the initial condition $S(0) = 90$. Moreover, the tank will become full at $t = 30$. Rewriting in linear normal form, we obtain

$$\frac{\mathrm{d}S}{\mathrm{d}t} + \left(\frac{3}{90+t}\right)S = 8,$$

and so we find the antiderivative

$$\int \frac{3}{90+t}\,\mathrm{d}t = 3\ln(90+t).$$

Hence, multiplying by the integrating factor $e^{3\ln(90+t)} = (90+t)^3$, we can rewrite the equation in the integrating factor form

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[(90+t)^3 S\right] = 8\cdot(90+t)^3$$

and hence

$$S(t) = \frac{1}{(90+t)^3}\int 8\cdot(90+t)^3\,\mathrm{d}t = \frac{1}{(90+t)^3}\left[2\cdot(90+t)^4 + C\right] = 2\cdot(90+t) + \frac{C}{(90+t)^3}.$$

To enforce the initial condition $S(0) = 90$, we find $C = -90^4$ and hence $S(30) \approx 202.031$ lb.

References

[1] C. H. Edwards, D. E. Penney, and D. T. Calvis. *Differential Equations and Linear Algebra*. Pearson, Upper Saddle River, NJ, 4 edition, Jan. 2017.